Achieving the Skinny Animal[1]

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Dedicated to Professor Sir William Hawthorne, Master, Churchill College

Abstract

The summary of my animal achievement games (generalized Tictactoe) in the Mathematical Games column of the Scientific American for April 1979 contains some wrong information which was supplied by me to Martin Gardner. We present the correct results here, and show in particular that the Skinny animal has board number 7 and move number 6.

1. History

One day a biologist walked into my office at the University of Michigan and asked if I could help him with a combinatorial problem. He wanted to investigated a theory of evolution for very small animals, beginning with one cell and growing just one cell at a time. He decided that it would be necessary to oversimplify the shape of the cell deliberately and chose square cells. He then drew the animals shown in Figure 1, (which I later named as indicated, with a little bit of help from my friend and colleague Ronald C. Read who said, "I don't know what to call the others, but I should call this one Fatty "). After we quickly constructed the 5-cell animals he was content and departed without leaving his name. The book [3] by Golomb called these animals 'polyominoes'. Several open questions concerning their enumeration were presented in [4] and [8].

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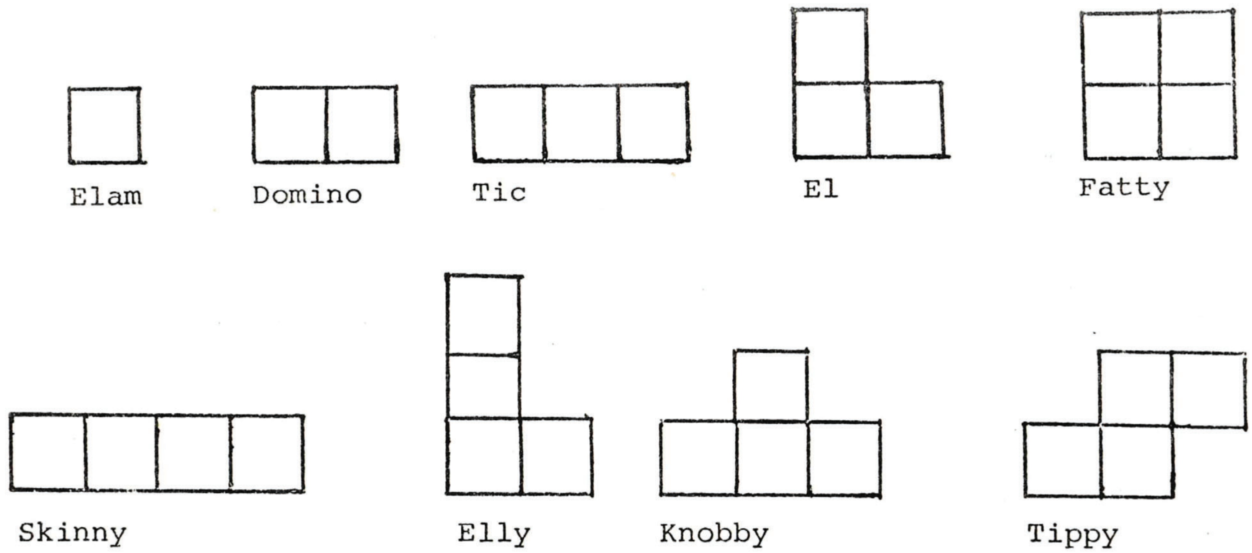

Figure 1. The smallest animals, with names.

When I first proposed achievement and avoidance games for graphs in [5], I did not realize that this would soon lead to generalizations of the traditional game of Naughts and Crosses (called Tictactoe in the United States)

2. Achieving Animals

Let A be a given animal and consider a b x b playing board. We adopt the convention that the first player to move is called Oh and that his moves are labelled $O_1, O_2, \ldots$ while the second player, Ex, makes moves $X_1, X_2, \ldots$ The winner, if any, is the player who first completes a copy of the objective animal A using only his own squares marked O or X. This is called the game of achieving animal A on a b^2 board; more briefly it is the achievement game (A,b). Note that any rotation or reflection of A is still regarded as animal A.

We illustrate an achievement game with the animal Elly of Figure 1. It is obvious that neither player can achieve Elly on a 3 x 3 board as it is very well known that the subanimal Tic cannot be achieved on a 3^2 board. Hence we now consider the achievement game(Elly, 4). In the 4^2 board of Figure 2, Oh makes a rational first move O_1 in any of the four central squares. Player Ex replies with X_1 in a central square adjacent to O_1.

Then Oh puts O_2 in the other central square next to O_1 and Ex realizes that he/she is in trouble. For no matter which end of the O_1-O_2 domino Ex picks, Oh will complete a Tic, as in Figure 2, by moving at the other end of the domino, and thus create a double threat in the two squares marked θ. So Ex resigns.

θ	O_3	θ	
	O_2		
	O_1	X_1	
	X_2		

Figure 2. Achieving Elly.

We say that the <u>board number</u> of Elly, written b(Elly), is 4 since Oh can complete an Elly on a 4^2 board but not on a 3^2 board. The <u>move number</u> of Elly, written m(Elly), is the smallest possible number of moves with which Oh can make Elly on this 4^2 board. Thus b(A) = m(A) = 4 when animal A is Elly.

3. Achieving Skinny

Based on the information which I supplied hastily to Martin Gardner for his annual April Fool's column in 1979, he reported in [2] that when animal A is Skinny, b(A) = 6 and m(A) = 8. However I was wrong about both the board number and the move number of Skinny! Finally it became patently clear that only an exhaustive check of all the possibilities would constitute a genuine proof. The correct values will now be derived.

A conventional labeling of the squares of a b^2 board Figure 3 uses letters a,b,... for the rows and numbers 1,2,...for the columns.

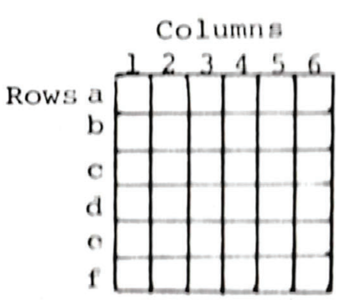

Figure 3. Notation for the squares of a board.

Theorem When A is the Skinny Animal, b(A) = 7 and m(A) = 6.
Proof It takes only a moment to verify that Oh cannot achieve Skinny on a 4^2 board or a 5^2 board. Figure 4 shows a 6^2 board in which the four central squares are marked C, the twelve middle squares M, and the outer squares are unmarked.

	M	M	M	M	
	M	C	C	M	
	M	C	C	M	
	M	M	M	M	

Figure 4. The central and middle squares of a 6^2 board.

Clearly the best opening move for Oh is to put O_1 into a C - square. Now there are three possibilites. If Ex decides to move X_1 in another C - square, Oh can win in only six moves as illustrated by the following game.

move	1	2	3	4	5
Oh	c3	d3	e2	d2	c2
Ex	c4	b3	e3	d4	resigns

If X_1 is placed in any of the 20 outer squares, the move is essentially wasted and Oh again proceeds to win in just six moves as in the game above.

But if Ex puts X_1 in any one of the twelve M - squares then Oh will not succeed in achieving Skinny and the result will be a draw! For after a few moves, Oh will make a threat which Ex can defend with a counterthreat, thus taking the attacking momentum from Oh. (Actually if Oh continues attacking now, he may enable Ex to win.) Thus with rational play by Ex, Oh cannot achieve Skinny on a 6^2 board.

Turning to a 7^2 - board, Oh places O_1 in the unique central square. No matter where X_1 is now located, O_2 is marked next to O_1 in a general direction"away" from X_1. The 7^2 - board now provides enough space for Oh to force Skinny in six moves as in the illustrated game above.

Thus when A = Skinny, we have b(A) = 7 and m(A) = 6.

4. Appendix

We conclude with nine brief observations concerning animal achievement games.

I. Fatty is not a winner.
It was shown in [2] how to tile an infinite plane with dominoes to form a blocking pattern for Fatty by having Ex reply to each Oh - move by completing the domino begun by Oh. As any Fatty in the plane must contain a whole domino, Oh cannot win.

II. An animal A is a minimal nonwinner
if Oh cannot achieve A, but any proper subanimal obtained from A is a winner. Including Fatty, 12 minimal nonwinners were displayed in [2] together with blocking patterns for each(including three patterns which need to be permuted in order to block the animals indicated). Of course any superanimal of a minimal nonwinner cannot be a winner either.

III. There are now 11 animals which have been proved to be winners. Using the letter - names (following Golomb [3]) of the three animals shown in Figure 5, their board and nove members are listed in Table 1.

Figure 3. Three more winners.

Table 1. The board and more numbers of the proved winners.

A	b	m
Elam	1	1
Domino	2	2
Tic	4	3
El	3	3
Skinny	7	6
Elly	4	4
Knobby	5	4
Tippy	3	5
"L"	7	7
"Y"	7	6
"Z"	6	6

The results for Skinny and the three letter - animals correct those listed in [2].

IV. There is exactly one animal which has not yet been proved to be a winner or nonwinner: Snaky (so named by Martin Gardner), shown in Figure 6. When I proposed the phrase "True Conjecture" for a conjecture about which one feels extremely confident, I did not know that Euler himself had already used it a few centuries earlier.

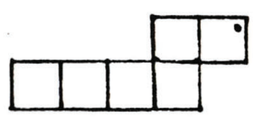

Figure 6. Snaky

True Conjecture Snaky is a winner.

My plausibility consideration for this assertion is that my former student, Dr. Geoffrey Exoo, won his last twenty games of Snaky achievement played on a sheet of "graph paper". However, it still remains to prove this and to determine the exact values of b and then m for Snaky.

After this conjecture has been proved, perhaps by an exhaustive computer programme, we will know that:

(a) There are exactly 12 winning animals;

(b) There are exactly 12 minimal nonwinners, each of which is blocked by a domino tiling pattern. (The symmetric appearance of this result would be quite pleasing.)

V. Let c(A) be the number of cells in animal A. Then A is called economical if A is a winner and m(A) = c(A).

Theorem There are exactly six economical animals: Elam, Domino, Tic, El, Elly and Tippy.

VI. It was shown in [6] that on a toroidal 5^2- board, the move number of Skinny is 8. The toroidal board and move numbers of the other animals are being invesigated.

VII. For the achievement game of "4 in a row" including diagonals as well as Skinny, the board and move numbers are both 5.

VIII. A fascinating book by Berlekamp, Conway and Guy [1] on mathematical games, presenting many new results, is about to appear.

IX. Many new achievement and avoidance games will be presented in the book [7] which is now being actively written.

ACKNOWLEDGEMENT

The corrected results for Skinny were achieved with the help of incisive comments by chess experts Adam Feinstein, David Levy and Kevin O'Connell.